Project Report

SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS USING HAAR WAVELET METHOD AND ANALYSIS OF STABILITY OF LINEAR AND NON LINEAR AUTONOMOUS DYNAMICAL SYSTEMS

by

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We thank you yet again.

Introduction

Wavelet analysis is a relatively new area in mathematic research. Wavelet analysis mainly includes the expression of functions. Functions are expanded to summation of “basic functions”, and every “basic function” is achieved by compression and translation of a mother wavelet function with good properties of smoothness and locality, which makes people study the properties of integer and locality in the process of expressing functions.

Fractional differential equations are generalised from integer order ones, which are obtained by replacing integer order derivatives by fractional ones. Compared with integer order differential equations, the fractional differential equations show many advantages over the simulation of natural physical process and dynamic system more accurate.They were found that various, especially interdisciplinary applications can be elegantly modelled with the help of the fractional derivatives. For examples, the nonlinear oscillation of earthquake can be modelled with fractional derivatives , and the fluid dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow.

Dynamical Systems deals with a system of differential equations that collectively describe the time dependence of a point in space. The stability of Dynamical systems has been explored in detail over the course of this project, using multiple methods.
Dynamical Systems find a wide array of application ranging from modelling the motion of a freely falling body to modelling diseases. It is only in recent years that the full potential of dynamical systems has been realised and immense inter disciplinary work is going into this field.

The aim of this project is to present the methodology to solve a set of fractional differential equations using the Haar wavelets method as well as discuss the stability of both linear and nonlinear dynamical systems.

Contents

1.1 Definition of fractional derivatives and integrals

(1) Riemann-Liouville definition : [1]

$$D_t^{\alpha} u(t) = \begin{cases} \dfrac{d^m u(t)}{dt^m} & \alpha = m \in \mathbf{N}; \\ \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{u(T)}{(t-T)^{\alpha-m+1}} \, dT & 0 \le m-1 < \alpha < m \end{cases}$$

$$J^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-T)^{\alpha-1} u(T) \, dT, \qquad t > 0, \ \text{with } J^0 u(t) = u(t)$$

(2) Caputo definition : [1]

$$D_*^{\alpha} u(t) = \begin{cases} \dfrac{d^m u(t)}{dt^m} & \alpha = m \in \mathbf{N}; \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{u^{(m)}(T)}{(t-T)^{\alpha-m+1}} \, dT & 0 \le m-1 < \alpha < m \end{cases}$$

$$D_*^{\alpha} J^{\alpha} u(t) = u(t)$$

$$J^{\alpha} D_*^{\alpha} u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{t^k}{k!}$$

1.2 Haar wavelets and function approximation

The Haar wavelet is the function defined on the real line **R** as that : [1]

$$H(t) = \begin{cases} 1, & 0 \le t < 0.5 \\ -1, & 0.5 \le t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

for $i = 1, 2, \ldots$, write $i = 2^j + k$ with $j = 0, 1, \ldots, 2^j - 1$. For $\forall$ t on [0,1], define $h_i(t) = 2^{j/2} H(2^j t + k)$, where $h_0(t) = 1$. Further the sequence $\{h_i\}_{i=0}^{\infty}$ is a complete orthonormal system in $L^2[0,1]$ and for $u \in C[0,1]$, series $\Sigma_i \langle u, h_i \rangle h_i$ converges uniformly to u, where $\langle u, h_i \rangle = \int_0^1 u(t) h_i(t) \, dt$

A function u(t) over the interval [0,1] can be decomposed as follows :

$$u(t) = \sum_{i=0}^{\infty} c_i h_i(t) \, dt \qquad \text{, where } c_i = \langle u(t), h_i(t) \rangle.$$

In computation practise, only the first k terms of above equation are considered, where k is power of 2. So we have :

$$u(t) \approx u_k(t) = \sum_{i=0}^{k-1} c_i h_i(t) \, dt$$

1.2.1. Example 1

Let $u(t) = 2t^2 - 3t + 1$, for a given value of k, Haar coefficients are calculated as $c_i = \langle u, h_i \rangle = \int_0^1 u(t) h_i(t) \, dt$, where i=1, 2, .., k-1 then taking $[c_i]$ as a row matrix, multiplying it with the Haar matrix of order k, we get an approximation of u(t) in every k subintervals of [0,1] .

For example, $H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$

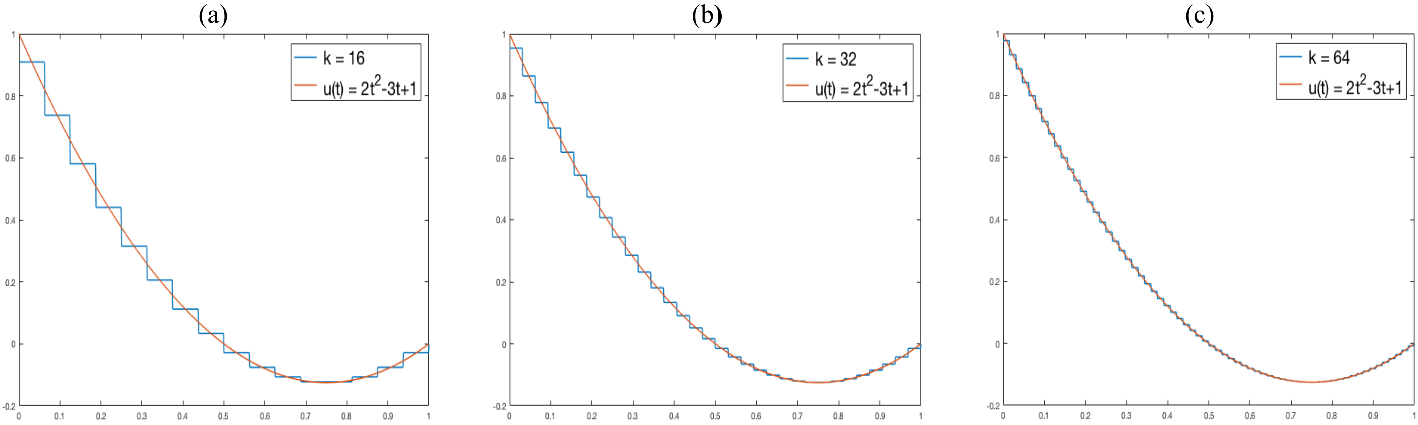

Figure 1.1 : Haar wavelets approximation of $u(t) = 2t^2 - 3t + 1$, in [0,1] for (a) k=16 , (b) k=32 , (c) k=64

1.3. Operational matrix of fractional integration of Haar wavelets [1]

Let $H_k(t) = [\, h_0(t), h_1(t), \ldots, h_{k-1}(t)\,]^T$ take the points $t_j = \frac{j - 1/2}{k}$, j = 1, 2, … k, then we define

$$H_{kxk} = \left[H_k\left(\frac{1}{2k}\right) \;\; H_k\left(\frac{1}{2k}\right) \;\; \ldots \;\; H_k\left(\frac{2k-1}{2k}\right) \right]$$

If I^α is fractional integration operator of Haar wavelets, we can get :

$(I^\alpha H_k)(t) = P^\alpha_{kxk} H_k(t)$, where P^α_{kxk} is called operational matrix of fractional integration of Haar wavelets.

Also, $P^\alpha_{kxk} = H_{kxk} F\alpha H^{-1}_{kxk}$

where,

$$F^\alpha = \frac{1}{k^\alpha} \frac{1}{\Gamma(2+\alpha)} \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{k-1} \\ 0 & 1 & \varepsilon_1 & \cdots & \varepsilon_{k-2} \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}$$

$$\varepsilon_i = (1+i)^{\alpha+1} - 2(i)^{\alpha+1} + (i-1)^{\alpha+1}$$

1.4. Fractional differential equation [1]

Now consider the following fractional differential equation with variable coefficients :

$$D^\alpha u(t) + a(t)\, u(t) = f(t)$$

where $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, m-1$, $0 < m-1 < \alpha \le m$

By approximating the function, $D^\alpha u(t)$ and $u(t)$, we have :

$$D^\alpha u(t) \approx D^\alpha u_k(t) = \sum_{i=0}^{k-1} c_i h_i(t)\, dt = c^T H_k(t)$$

$$u(t) \approx u_k(t) = c^T P^\alpha H_k(t)$$

where $c = [c_0, c_1, \ldots, c_{k-1}]^T$

Substituting we get, $c^T H_k(t) + a(t)\, c^T P^\alpha_{kxk} H_k(t) = f(t)$

Co-efficient a(t) can be dispersed into $a(t_i)$ and f(t) into $f(t_i)$ (i = 0, 1, 2…, k-1) and converted into matrix form of desired dimensions.

$$
A_i = \begin{bmatrix} a(t_0) & 0 & 0 & \cdots & 0 \\ 0 & a(t_1) & 0 & & \vdots \\ 0 & 0 & a(t_2) & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & a(t_{k-1}) \end{bmatrix} \qquad f = [\, f(t_0)\, ,\, f(t_0)\, ,\, \ldots\, ,\, f(t_{k-1})\,]^T
$$

Therefore, we get $c^T H(t) + c^T P^{\alpha}_{kxk} H(t) A = f^T$

The above equation is a linear system of algebraic equation, where $H = [\, H_k(t_0)\, ,\, H_k(t_1)\, ,\, \ldots\, ,\, H_k(t_{k-1})\,]$.

1.5. Generalised Fractional differential equation

To generalise, let us consider the following fractional differential equation with variable coefficient :

$$a_1(t)\ D_*^{\alpha_1} u(t) + a_2(t)\ D_*^{\alpha_2} u(t) + \ldots + a_s(t)\ D_*^{\alpha_s} u(t) = f(t)$$

Here consider $\alpha = \max_{1 \le i \le s} \{ \alpha_i \}$

$$u^{(i)}(0) = u_{i,init}\ , \quad i = 0,1,\ldots,q-1, \quad 0 < q-1 < \alpha \le q$$

where q is an integer

Methodology :

Let

$$\varphi_{mxm} = \left[H_m\left(\frac{1}{2m}\right) H_m\left(\frac{3}{2m}\right) \ldots\ H_m\left(\frac{2m-1}{2m}\right) \right]$$

where $H_m(t) = [\, h_0, h_1, h_2, \ldots , h_{m-1}]^T$ and

$$h_i(t) = \begin{cases} 1 & \frac{k-1}{2j} \le t < \frac{k-0.5}{2j} \\ -1 & \frac{k-0.5}{2j} \le t < \frac{k}{2j} \\ 0 & \text{otherwise} \end{cases}$$

for i = 1, 2, … , m-1 where $m=2^M$, $M \in \mathbf{Z}$, j and k are integer decomposition of i , i.e. $i = 2^j + k - 1, \quad 0 \le j < i$ and $1 \le k < 2^j + 1$

$$h_0(t) = 1$$

Calculate the operational matrix P^{α}_{mxm} for the given m, and α as :

$$P^{\alpha}_{mxm} = \varphi_{mxm} F^{\alpha}\ \varphi^{-1}_{mxm}$$

where,

$$F^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(2+\alpha)} \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{m-1} \\ 0 & 1 & \varepsilon_1 & \cdots & \varepsilon_{m-2} \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix}$$

$$\varepsilon_i = (1+i)^{\alpha+1} - 2(i)^{\alpha+1} + (i-1)^{\alpha+1}$$

Now approximating, $D_*^{\alpha_i} u(t)$ as :

$$D_*^{\alpha_i} u(t) \approx D_*^{\alpha_i} u_m(t) = c^T P^{\alpha-\alpha_i} H_m(t) + \sum_{r=1}^{\alpha-\alpha_i} u_{\alpha-r,init}\, [1\ 1\ 1 \ldots 1]\ \ \varphi^{-1}_{mxm} P^{\alpha-\alpha_i-r} H_m(t)$$

Similarly we can disperse $a_i(t)$ into $a_i(t_j)$ and $f(t)$ into $f(t_j)$ ($j = 0, 1, 2\ldots, m-1$) and converted into matrix form of desired dimensions.

$$A_i = \begin{bmatrix} a_i(t_0) & 0 & 0 & \cdots & 0 \\ 0 & a_i(t_1) & 0 & & \vdots \\ 0 & 0 & a_i(t_2) & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & a_i(t_{m-1}) \end{bmatrix} \qquad f = [\, f(t_0)\,,\, f(t_0)\,, \ldots ,\, f(t_{m-1})\,]^T$$

Substituting we get a linear system of m algebraic equation of m variables . From these equation we get the Haar-coefficients of u(t).

Finally we get an approximation of u(t) by this expression.

$$u(t) \approx u_m(t) = c^T P^{\alpha} H_m(t) + \sum_{r=1}^{\alpha} u_{\alpha-r,init} [1\ 1\ 1\ldots1] \quad \varphi^{-1}_{mxm} P^{\alpha-\alpha_i-r} H_m(t)$$

1.5.1. Example 2

Let us consider the following fractional differential equation :

$$t^4 D^4_* u(t) - \left(\sin t + \frac{t^2}{2}\right) D^3_* u(t) + D^{\alpha}_* u(t) - 24\, u(t) = \frac{3\,t^2}{2} + (3 - 24\,t) \sin t - \frac{3}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)} t^{4-\alpha}$$

where $0 < \alpha < 1$, with the initial conditions as :

$$u(0) = 0\,,\ u^{(1)}(0) = 0\,,\ u^{(2)}(0) = 0\,,\ u^{(3)}(0) = -3$$

Solution : The exact solution of $u(t) = t^4 - \frac{t^3}{2}$ the linear system of algebraic equations were solved by the function FSOLVE in MATLAB for various values of m.

(a) (b) (c)

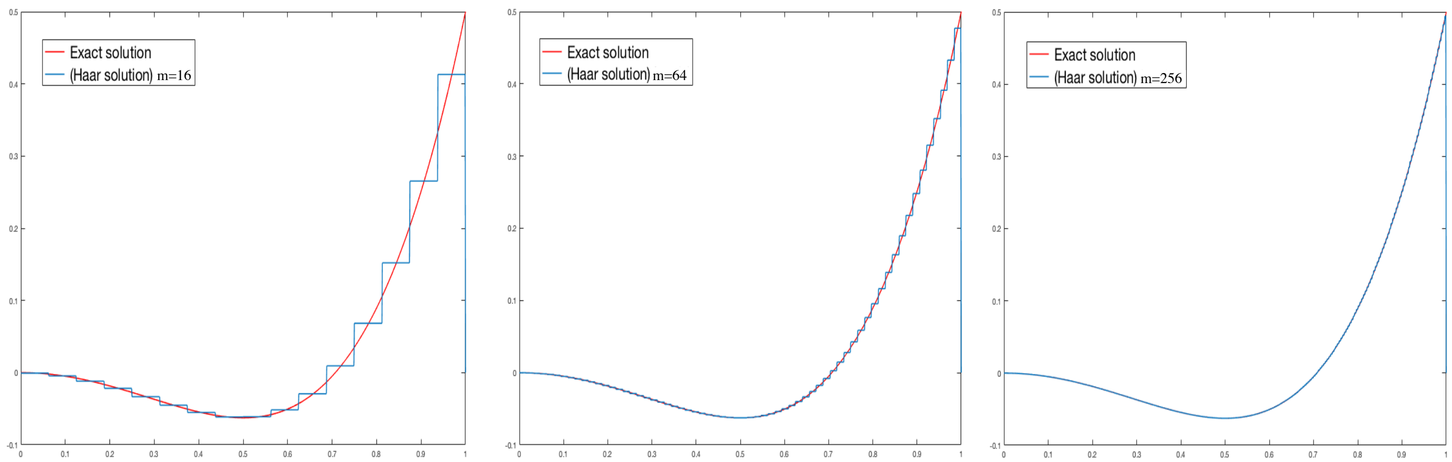

Figure 1.2 : Haar solution of the Differential equation (7.1 Example), in [0,1] for (a) m=16 , (b) m=64 , (c) m=256

As m increases, the error decreases between exact solution and the solution derived by Haar wavelets approximation as discussed in section 1.6. **[1]**

1.5.2. Solving non linear fractional differential equation

As usual, every component of the differential equation is converted into their matrix forms. Component multiplication in the matrix form is done by Hadamard product of matrixes (element wise multiplication of matrix)

For example, the Hadamard product for a 3x3 matrix A with a 3x3 matrix B is :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix}$$

Further using FSOLVE to solve this non-linear system of equation, the Haar coefficients are calculated. (Recommended algorithm : Levenberg marquardt algorithm)

1.5.3. Changing interval of solution

To change the interval of solution from [0, 1] to [0, b] where b ∈ R, the operational matrix and definition of Haar matrix gets modified :

1.5.3.1. Operational matrix P^{α}_{mxm} for a given m is calculated as :

$$P^{\alpha}_{mxm} = \varphi_{mxm} F^{\alpha} \varphi^{-1}_{mxm}$$

where ,

$$F^{\alpha} = \frac{b^{\alpha}}{m^{\alpha}} \frac{1}{\Gamma(2+\alpha)} \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{m-1} \\ 0 & 1 & \varepsilon_1 & \cdots & \varepsilon_{m-2} \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}$$

$$\varepsilon_i = (1+i)^{\alpha+1} - 2(i)^{\alpha+1} + (i-1)^{\alpha+1}$$

1.5.3.2. Haar matrix is defined as :

$$\varphi_{mxm} = \left[H_m\left(\frac{b}{2m}\right) \; H_m\left(\frac{b}{2m}\right) \; H_m\left(\frac{b}{2m}\right) \cdots \; H_m\left(\frac{(2m-1)\,b}{2m}\right) \right]$$

where $H_m(t) = [h_0, h_1, h_2, \ldots, h_{m-1}]^T$ and

$$h_i(t) = \begin{cases} 1 & \frac{k-1}{2j} \le \frac{t}{b} < \frac{k-0.5}{2j} \\ -1 & \frac{k-0.5}{2j} \le \frac{t}{b} < \frac{k}{2j} \\ 0 & \text{otherwise} \end{cases}$$

$$h_0(t) = 1$$

for i = 1, 2, … , m-1 where $m = 2^M$, M ∈ **Z** , j and k are integer decomposition of i , i.e.

$i = 2^j + k - 1, \quad 0 \le j < i$ and $1 \le k < 2^j + 1$

1.5.4. System of non-linear fractional differential equation

Let us consider a set of n independent fractional differential equations constituting of n unknown functions. In this case, each fractional differential equation gets converted into m equations (where m is the order of approximation), so in all the set of fractional D.Es gets converted into m·n number of equations which gives the m·n number of Haar-coefficients (m coefficients for each unknown function). Hence, solving the problem.

1.5.5. Example 3

Let us consider the set of equations :

1. $t\, D_*^1 u(t) - y(t) + y(t)\, u(t) = 0$
2. $D_*^1 u(t)\; D_*^1 y(t) + u(t) = 1$

where $u(0) = 0$, $y(0) = 0$

The exact solution of the above set of equation is $u(t) = t$ and $y(t) = 1 - e^{-t}$. Using Haar wavelets to solve the set of non-linear equations in the interval [0 , 2], we plot Haar approximated solution of u(t) and y(t) using FSOLVE for the various values of m.

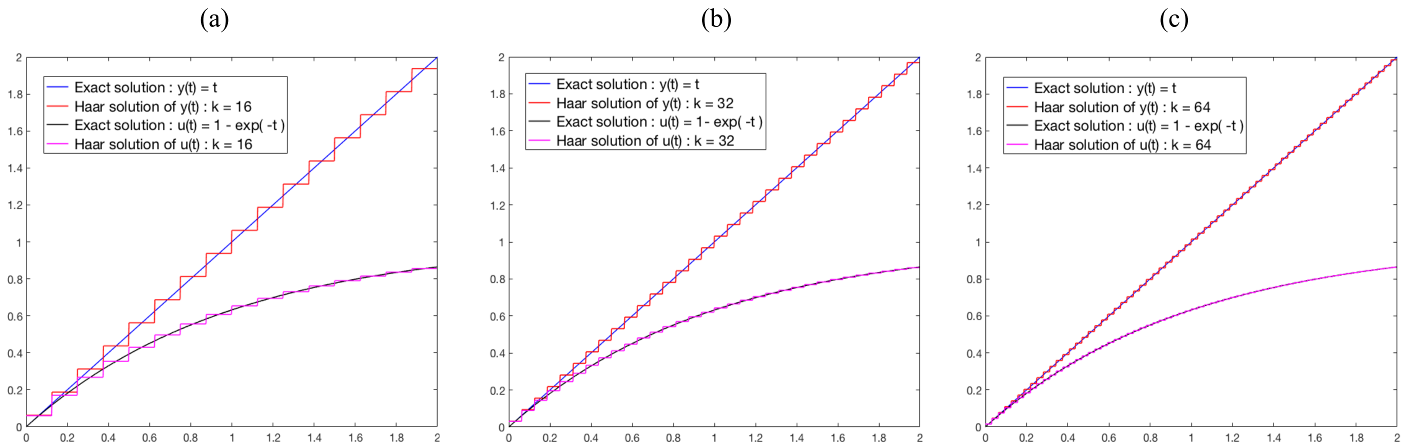

Figure 1.3 : Haar solution of the Differential equation (1.5.1 Example), in [0,1] for (a) m=16 , (b) m=32 , (c) m=64

1.6. Error analysis [1]

Suppose that the function $u_k(t)$ obtained by using Haar wavelets are the approximation of u(t), then we have an exact upper bound as follows :

$$\| u(t) - u_k(t) \|_E \leq \frac{M}{\Gamma(\alpha)\, \Gamma(m-\alpha)\, \alpha\, (m-\alpha)\, [1 - 2^{2(\alpha-m)}]^{1/2}\, k^{(m-\alpha)}}$$

$$\text{where } \| u(t) \|_E = \left(\int_0^1 u^2(t)\, dt \right)^{1/2}$$

Here M is estimated as $\max_{1 \le i \le k} \left| u^{(m)}(t_i) \right|$ where $t_j = \frac{j - 1/2}{k}$, j = 1, 2, …k, and m is the integer upper bound of the order of the DE.

A conclusion is drawn that Haar wavelets method is convergent when it is used to solve the numerical solution of fractional differential equations. Because it is clear that as $k \to \infty$, $\| u(t) - u_k(t) \|_E \to 0$.

2.1 Dynamical Systems

Definition:

A Dynamical System is a mathematical formulation of any rule that describes the time dependence of a point in space.

More formally, A smooth dynamical system on $\Re^n$ is a continuously differentiable function $\phi : \Re \times \Re^n \to \Re^n$ $\phi(t, x) = \phi_t(x)$
1) $\phi_0 : \Re^n \to \Re^n$ is the identity function of $\phi_0(X_o) = X_o$
2)$\phi_t o \phi_s = \phi_{t+s}$

2.1.1 Example :
$x^{'} = ax$
Here, $\phi_t(x_0) = x_0 e^{at}$ which is actually the solution on the system.
$\phi_0(X_0) = X_0 e^0 = X_0$
and, $\phi_{t+s}(X_0) = e^{(t+s)a} = e^{ta}e^{sa} = \phi_t(X_0)\phi_s(X_0)$

2.2 Linear Dynamical Systems

2.1.1 Definition:

A Linear Dynamical system can be thought of as a system in which no term involved has a non-linear(i.e. degree>1 or product of two variables) terms involved.
Planar Linear Dynamical Systems are basically 2-D Linear Dynamical Systems.

2.2.2 Example :
$x^{'} = ax + by$
$y^{'} = cx + dy$ where a,b,c,d $\in \Re$

2.3 Discussion of equilibrium points on Planar Linear Dynamical Systems

2.3.1 Definition of an Equilibrium point:

An equilibrium point is often understood as the point at which the rate of change is zero. That is in effect the first order derivative which represents rate of change is zero. It is also commonly referred to as a fixed point. For a system of course, the rate of change of each variable involved should be equated to 0 and the common point of intersection obtained is in fact the equilibrium point.

2.3.2 Example:
$x^{'} = x$
$y^{'} = x + y$

We equate $x^{'} = 0$, we get

$$x = 0 \tag{1}$$

and we also simultaneously equate $y^{'} = 0$, we get

$$x + y = 0 \tag{2}$$

on solving 1 and 2 we get the equilibrium point as x=0 and y=0

It is easy to note that every planar linear system has (0,0) i.e. the origin as an equilibrium point.

For a Linear system,

$$x^{'} = ax + by$$
$$y^{'} = cx + dy$$

where $a, b, c, d \in \Re$

We define

A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Using this matrix A, we can rewrite the dynamical system as

$$X^{'} = AX \tag{3}$$

As mentioned earlier, the system (3) has origin as an equilibrium point. Further, if det(A)=0 there will be more than one equilibrium point.

2.4 Classification of equilibrium points in Planar Linear Dynamical Systems

From Linear Algebra we know how to calculate eigenvalues and eigenvectors of a matrix A. Using this knowledge we can solve a planar system with ease.

The three possible cases are:

Case 1: Two distinct and real eigenvalues (λ_1 and λ_2)

On solving (3),

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2 \tag{4}$$

X(t) is the solution of the system, α and β are real constants and V_1 and V_2 are eigenvectors corresponding to λ_1 and λ_2 respectively.

a) Both eigenvalues are negative

Let us assume for simplicity that the matrix corresponding to the system (3) is A= $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

(4) becomes,

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2 \tag{5}$$

V$_1$ = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now as $t \to \infty$ we can see that x(t) and y(t) both tend to 0 asymptotically. This equilibrium is stable as the solution as a whole i.e. $X(t) \to 0$ and is often referred to as a sink.

b) Both eigenvalues are positive

Let us assume for simplicity that the matrix corresponding to the system (3) is

A= $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

(4) becomes,

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2 \tag{6}$$

V$_1$ = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now as $t \to \infty$ we can see that x(t) and y(t) both tend to ∞ as they are both increasing. This equilibrium is clearly unstable as the solution as a whole i.e. $X(t) \to \infty$ and is often referred to as a source.

Case 2: Complex eigenvalues

a) Matrix is of the form: A= $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$

Here we get the General solution as:

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 \tag{7}$$

$V_1 = \begin{pmatrix} cos(\beta t) \\ -sin(\beta t) \end{pmatrix}$ $V_2 = \begin{pmatrix} sin(\beta t) \\ cos(\beta t) \end{pmatrix}$

Each of these solutions are periodic with period $2\pi/\beta$. This system formed is called a center.

b) Matrix is of the form: A= $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Here we get the General solution as:

$$X(t) = e^{\alpha t} V_1 + e^{\alpha t} V_2 \tag{8}$$

$V_1 = \begin{pmatrix} cos(\beta t) \\ -sin(\beta t) \end{pmatrix}$ $V_2 = \begin{pmatrix} sin(\beta t) \\ cos(\beta t) \end{pmatrix}$

These solutions would represent a center if α were to be zero. The $e^{\alpha t}$ term makes the system form a spiral. The spiral spins away from (0,0) if $\alpha > 0$ and is an unstable configuration, called a spiral source. The spiral spins towards the origin if $\alpha < 0$ and is a stable configuration, called a spiral sink.

Case 3: Repeated eigenvalues
Let us consider the case where the common eigenvalue yields only a single linear independent eigenvector.
Matrix is of the form: A= $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

The only linear independent eigenvector is (0,1) and the general solution is of the form

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2 \tag{9}$$

$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} t \\ 1 \end{pmatrix}$

It can clearly be seen that if $\lambda > 0$ and as $t \to \infty$, X(t) tends to (0,0). This is a stable configuration. However, if $\lambda < 0$ and as $t \to \infty$, X(t) tends away from the origin forming an unstable configuration.

2.5 Classification of equilibrium points using the Trace-Determinant method

Let us consider the dynamical system

$$X' = AX$$

where the corresponding matrix is given by A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let T denote the Trace(A), T=a+c
and D denote the determinant, D=ad-bc

The eigenvalues are in fact the roots of the characteristic equation which comes from det(A-λI) = 0

They can be represented as $\lambda_1 = \frac{T+\sqrt{T^2-4D}}{2}$; $\lambda_2 = \frac{T-\sqrt{T^2-4D}}{2}$

Based on this we can plot the Trace-Determinant Plane

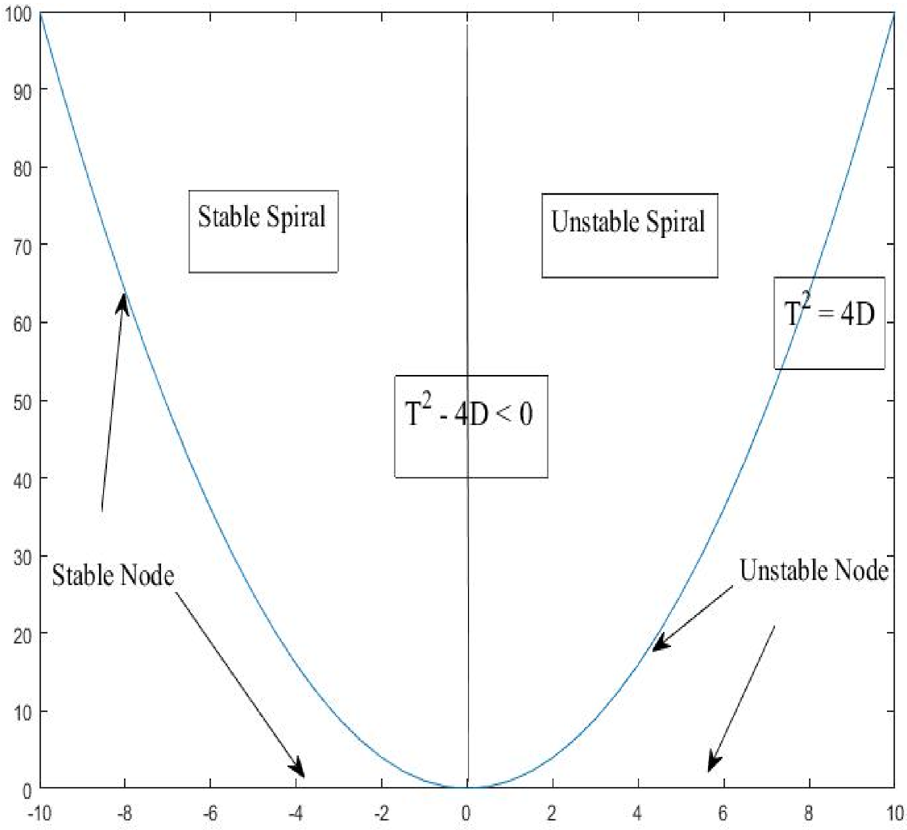

Figure 2.1: The Trace-Determinant Plane

2.6 Equilibrium of Non-Linear Systems using Linearization

We can use the fact that for systems having an equilibrium point near the origin terms like x^2, y^3, etc. can be neglected. However this process may not always work.

2.6.1 Example:

$x' = x + y^2$
$y' = -y$

We equate $x' = 0$ and $y' = 0$, we get the equilibrium point as x=0 and y=0. Now if we were to rewrite the system by neglecting y^2

$x' = x$
$y' = -y$

We still get the equilibrium point as x=0 and y=0. And solving will show us that both give the same answer.

2.6.2 Example:

$x' = x^2$
$y' = -y$

We equate $x' = 0$ and $y' = 0$, we get the equilibrium point as x=0 and y=0. Now if we were to rewrite the system by neglecting x^2

$x' = 0$

$y^{'} = -y$

We see that for the new system; y=0 and x=constant i.e. the x axis is the set of equilibrium points. It is evident that the equilibrium points of the system has changed and therefore the linearization method fails here.

2.7 Equilibrium of Non-Linear Systems using Jacobian Matrix

Let us have the system as X$^{'} = g(X)$
Now on equating $X^{'} = 0$ say we are to get X_1 as an equilibrium point.
Let J denote the Jacobian Matrix of the system.

1) If $J(X_1)$ were to yield eigenvalues with negative real parts, we can conclude the X_1 is an asymptotically stable equilibrium point.
2) If $J(X_1)$ were to yield eigenvalues with positive real parts, we can conclude the X_1 is an unstable equilibrium point.
3) If $J(X_1)$ yields eigenvalues with both positive and negative real parts, we classify the equilibrium point as saddle point.
4) If $det(J) = 0$, this process if insufficient to comment on stability.

Let us look at Example 2 from the previous section. The system was
$x^{'} = x^2$
$y^{'} = -y$

and the Jacobian can be calculated as,

J= $\begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$

The Determinant of J at the equilibrium point (0,0) is zero and hence we can't comment on it's stability here.

2.8 Drawing a Phase Portrait to understand stability

Let us consider the dynamical system

$$X^{'} = AX$$

where the corresponding matrix is given by A= $\begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$

1)First, we find the general solution of the system.
X(t)=c$_1 e^t V_1 + c_2 e^{3t} V_2$

V$_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2) The Phase portrait is just a rough representation of the family of curves X(t). As is it difficult to plot all solutions corresponding to various values of c_1 and c_2, We plot the four basic ones. These correspond to $c_1 = +1, -1$ when $c_2 = 0$ and $c_2 = +1, -1$ when $c_1 = 0$.

3)These four correspond to the straight lines passing through the origin.
Now as $t \to \infty$, e^{3t} dominates over e^t. Further as $t \to -\infty$, e^t dominates over e^{3t}.
Therefore, a general solution starting from the origin would follow the trend of starting parallel to $\lambda = 1$ and end up becoming parallel to $\lambda = 3$.

The equilibrium point in clearly an unstable one as the solutions tend away from the origin. This can be verified from the phase portrait as well.

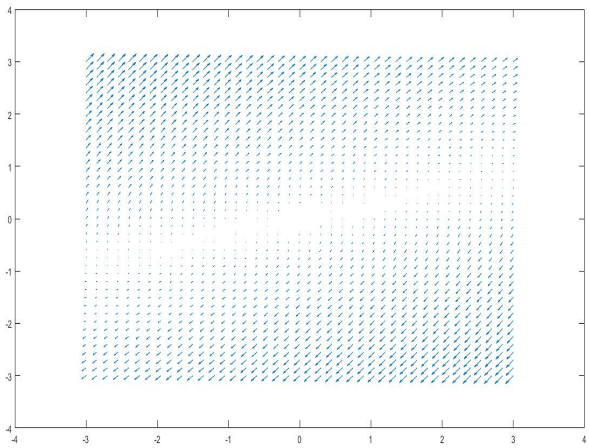

Figure 2.2: Phase Portrait corresponding to Matrix A

The main advantage of being able to plot the phase portrait is that is saves trouble in analysing the solutions and the graph very clearly shows the stability at the equilibrium point.

3 Conclusion

Over the duration of the project, we started from the basic definition of wavelets and then explored the various features of wavelets. We were particularly interested in some features that helped us in solving differential equations of fractional order. Fractional order differential equation we came to understand are a generalisation of integer order ordinary differential equations.
We started by focusing on the Haar Wavelets and studied various properties like scaling on the interval [0, 1), and various other properties. We used the haar wavelets as a set of orthonormal basis functions on [0, 1) to approximate various functions and then further used this technique of approximation to solve fractional order differential equations.

Apart from our work on wavelets we also studied in detail about dynamical systems. Stressing more on linear dynamical systems, we understood different methods to discuss stability.
While we started by focusing mainly on theoretically showing the existence of equilibrium points and classifying them we then move on to plotting phase portraits. Phase Plots help in visualising the phenomenon of stable and unstable equilibrium points.

MATLAB is a coding environment which proved essential throughout this project in understanding various concepts and plotting important graphs.
Implementation of various MATLAB functions to solve differential equations and plot phase portraits has made both the report and our project complete in many aspects.

4 References

[1] Yiming Chen, Mingxu Yi, Chunxiao Yu. Error Analysis for numerical solution of fractional differential equation by Haar Wavelets method, Journal of Computational Science 3 (2012) 367-373

[2] Naresh Berwal, Dinesh Panchal, C.L.Parihar. Solutions of differential equation based on Haar Operational Matrix, Palestine Journal of Mathematic Vol.3(2) (2014), 281-288

[3] G.Hariharan, K.Kanan. An overview of Haar Wavelet Method for Solving Differential and Integral Equations, 2013

[4] Dennis Zill, Warren Wright. Advanced Engineering Mathematics, $5^{th} edition$

[5] Morris Hirsch, Robert L.Devaney, Stephen Smale. Differential Equations, Dynamical Systems, and an Introduction to Chaos. Elsevier 3^{rd} edition

[6] William John Palm. Introduction to MATLAB for Engineers. 3^{rd} Edition.

[7] Wavelet Toolbox Documentation - MathWorks India.